UNCERTAINTY RELATIONS FOR ENTANGLED STATES

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A generalized uncertainty relation for an entangled pair of particles is obtained if we impose a symmetrization rule for all operators that we should employ when doing any calculation using the entangled wave function of the pair. This new relation reduces to Heisenberg's uncertainty relation when the particles have no correlation and suggests that we can have new lower bounds for the product of position and momentum dispersions.

Key words: entanglement, identical particles, uncertainty relations.

1   INTRODUCTION

In this letter we examine the derivation of the uncertainty relations for a pair of entangled particles motivated by the recent experiment of Kim and Shih [1], a realization of Popper's experiment [2], which can be considered an extension of the EPR argument [3].

Let us begin with a brief review of Popper's experiment, as done by Kim and Shih: The entangled pair of photons are produced by Spontaneous Parametric Down Conversion (SPDC). Kim and Shih's measurements are conditional in the sense that the detection of photon 2 by a y-scanning detector D_2 is coincidental with the detection, in detector D_1, of photon 1 after its passage through slit A. (See figure 1). Two cases are considered:

(a) Slit A and slit B (placed along the trajectory of photon 2) have the same width.
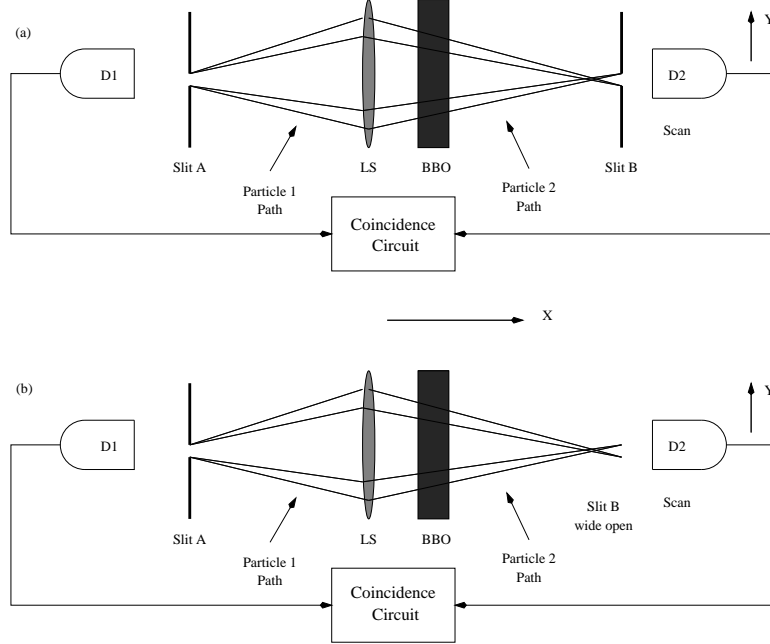
(b) Slit B is left wide open.

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Figure 1: This is almost the same figure given in Kim and Shih's paper. Part (a) represents the set-up with both slits with the same width. Part (b) represents slit B wide open. Beta Barium Borate (BBO) is the crystal where a laser beam produces by SPDC the entangled pair of photons. LS is the lens that produces a "ghost image" of slit A and localizes photon 2 when detecting photon 1 at slit A. See reference [1] for more details.

The first case does not present any challenge to our understanding as the Heisenberg uncertainty relation applies to both branches. The interesting situation comes from the second set-up, where Kim and Shih's experiment suggests that $\Delta y_2 \Delta p_{y_2} < \hbar$ in an apparent violation of Heisenberg relation. They explain the result invoking the necessity of working with the entangled wave function of the pair (biphoton wavefunction)[1].

Two recent papers [4, 5] have discussed this problem from yet two different points of view. The paper by Short [4] claims that there is no violation of the uncertainty principle and justifies this claim by affirming that the two photons:

> do not interact with each other after their initial creation and must evolve independently between measurements when they are space-like separated.

We do not agree with this assumption because we think an entangled pair of photons need not obey this. Short's main argument is that the observed coincident patterns are dominated by a blurring of the photons' path which he

considers intrinsic to the experimental set-up of Kim and Shih. However, we could expand the pump beam diameter and invalidate Short's analysis.

The second paper, by Unnikrishnan [5], approaches the problem in what seems to us the right but still incomplete way. Looking at the wave function of the entangled pair, without showing any of the actual calculations, Unnikrishnan claims that the constraint of momentum conservation explains Kim and Shih results. In the following we give a complete treatment of the problem calling attention to points that have passed unnoticed in the above mentioned papers.

2 OUR RESULTS

In agreement with [1,5] we recognize that when dealing with entangled systems we should not use wave functions that describe the isolated evolution of a member of the system, but rather we should use the entangled wave function of the system. These entangled wave functions have to obey the symmetrization rules of Quantum Mechanics: (anti-) symmetric wave functions for (fermions) bosons.

It is however less known, despite appearing in some text books [6], the fact that for a correlated system (entangled system) we must deal with what is called *physical observables*, which have to obey symmetry requirements as well. As shown by Cohen-Tannoudji [6], *physical observables* must commute with all the permutation operators that appear in the system.

We restrict ourselves to the case of a pair of correlated particles, but a generalization to a system of N entangled particles is straightforward.

Let us define the following operator:

$$\mathcal{O}(1,2) = \sum_{i=1}^{n} A_i(1) \otimes B_i(2), \tag{1}$$

where n is an integer greater than zero, not necessarily equal to the number of entangled particles (an example is the total angular momentum of two particles $J(1,2) = L(1) \otimes \mathcal{I}_2 + S(1) \otimes \mathcal{I}_2 + \mathcal{I}_1 \otimes L(2) + \mathcal{I}_1 \otimes S(2)$, where $L(i)$, $S(i)$ and $\mathcal{I}_i$ are the orbital, spin angular momentum and the identity operator of particle i). $A_i(1)$ and $B_i(2)$ can be any observables initially defined in the state spaces $\mathcal{E}(1)$ and $\mathcal{E}(2)$ of particles 1 and 2, and then extended into $\mathcal{E}(1,2)$, the state space of the two-particle system. The state space $\mathcal{E}(1,2)$ is the tensor product of the state spaces of particles 1 and 2, $\mathcal{E}(1,2) = \mathcal{E}(1) \otimes \mathcal{E}(2)$.

The operator $\mathcal{O}(1,2)$ is called a *physical observable* if it satisfies the following commutation relation:

$$[\mathcal{O}(1,2), P_{21}] = 0, \tag{2}$$

where P_{21} is the permutation operator in the state space $\mathcal{E}(1,2)$. It can be shown [6] that P_{21} is hermitian and obeys the following relation:

$$P_{21}\mathcal{O}(1,2)P_{21}^{\dagger} = \mathcal{O}(2,1). \tag{3}$$

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Using Eq. (3) and defining the extended position and momentum operators, in a given direction, $Q(1,2) = Q(1) \otimes \mathcal{I}_2 + \mathcal{I}_1 \otimes Q(2)$ and $P(1,2) = P(1) \otimes \mathcal{I}_2 + \mathcal{I}_1 \otimes P(2)$, where $\mathcal{I}_i$ is the identity operator in the state space of particle i, we can show that:

$$[Q(1,2), P_{21}] = [P(1,2), P_{21}] = 0. \tag{4}$$

Therefore, $Q(1,2)$ and $P(1,2)$ are *physical observables*.

Experimentally, in a coincidence measurement, $Q(1,2)$ is the sum of the positions of both particles and $P(1,2)$ gives the total momentum of the system in a given direction. We now require that when deducing the uncertainty relation for a correlated system we should use only *physical observables* as follows:

$$(\Delta Q(1,2))^2 (\Delta P(1,2))^2 \geq \frac{|\langle [Q(1,2), P(1,2)] \rangle|^2}{4}. \tag{5}$$

We cannot have the traditional relations

$$(\Delta Q(i))^2 (\Delta P(i))^2 \geq \frac{|\langle [Q(i), P(i)] \rangle|^2}{4}, \tag{6}$$

because $Q(i)$ and $P(i)$, where $i = 1$ or $i = 2$, are not *physical observables* (they do not commute with the permutation operator). Manipulating Eq. (5) we have (from now on we write $Q(i) = Q_i$ and $P(i) = P_i$ to simplify notation):

$$[(\Delta Q_1)^2 + (\Delta Q_2)^2 + 2(\langle Q_1 Q_2 \rangle - \langle Q_1 \rangle \langle Q_2 \rangle)] \times$$

$$[(\Delta P_1)^2 + (\Delta P_2)^2 + 2(\langle P_1 P_2 \rangle - \langle P_1 \rangle \langle P_2 \rangle)] \geq \hbar^2. \tag{7}$$

Assuming, as done by Popper and implictly by Kim and Shih, that $\Delta Q_1 = \Delta Q_2$ we get:

$$\left[(\Delta Q_2)^2 + (\langle Q_1 Q_2 \rangle - \langle Q_1 \rangle \langle Q_2 \rangle) \right] \times$$

$$\left[\frac{(\Delta P_1)^2}{2} + \frac{(\Delta P_2)^2}{2} + (\langle P_1 P_2 \rangle - \langle P_1 \rangle \langle P_2 \rangle) \right] \geq \frac{\hbar^2}{4}. \tag{8}$$

This last expression should be the correct uncertainty relation when treating a correlated pair of particles and not the naive Heisenberg uncertainty relation:

$$(\Delta Q_2)^2 (\Delta P_2)^2 \geq \frac{\hbar^2}{4}. \tag{9}$$

We suggest that Kim and Shih's experimental results should be analyzed using Eq. (8) instead of Eq. (9). We should mention that A. C. de la Torre, P. Catuogno and, S. Ferrando [7] had already obtained the functions $\langle P_1 P_2 \rangle - \langle P_1 \rangle \langle P_2 \rangle$ and $\langle Q_1 Q_2 \rangle - \langle Q_1 \rangle \langle Q_2 \rangle$, which they call Quantum Covariance Functions (QCF), in a quite different context. They had proved that the QCF vanishes if and only if the system is separable. This means that, as entanglement of a pair of identical particles means non-separability, the QCF does not

vanish in Eq. (8) and we do have an uncertainty relation that is different from Heisenberg's.

In order to illustrate what we have discussed up to this point we now study Eq. (8) for a particular 1-dimensional wave function:

$$\Psi(x_1, x_2, t) = \int f(k_1, k_2) \exp[i(k_1 x_1 + k_2 x_2 + \omega t)] dk_1 dk_2, \qquad (10)$$

where $\omega = \frac{E}{\hbar}$, E is the energy of the system, k_1 and k_2 are the wave numbers of particles 1 and 2 respectively and

$$f(k_1, k_2) = \frac{a}{(2\pi)^{3/2}} \left[\exp\left(\frac{-a^2}{4}(k_1 - k_0)^2\right) \exp\left(\frac{-a^2}{4}(k_2 + k_0)^2\right) \right] +$$

$$\frac{a}{(2\pi)^{3/2}} \left[\exp\left(\frac{-a^2}{4}(k_1 + k_0)^2\right) \exp\left(\frac{-a^2}{4}(k_2 - k_0)^2\right) \right]. \qquad (11)$$

Eq. (11) represents a symmetric correlated pair of particles where $k_1 + k_2 = 0$, that is, $k_1 = -k_2 = k_0$ or $k_1 = -k_2 = -k_0$. Setting $t = 0$ in Eq. (10) we show that for $k_0 x_1 \ll 1$ and $k_0 x_2 \ll 1$ that:

$$\langle x_1 \rangle = \langle x_2 \rangle = \langle p_1 \rangle = \langle p_2 \rangle = 0, \qquad (12)$$

$$\langle x_1 x_2 \rangle = \frac{k_0^2 a^4}{2}; \langle p_1 p_2 \rangle = -2\hbar^2 k_0^2. \qquad (13)$$

Eq. (13) shows that Eq. (8) should not be equal to the usual Heisenberg uncertainty relation.

3 CONCLUSION

We have suggested here that when dealing with entangled systems of identical particles we should use *physical observables* (those that commute with all the permutation operators of the system) in whatever calculations we perform using the (anti-) symmetric wave function of the system.

Applying the above assumption in the deduction of the uncertainty relation for a pair of entangled particles we have got (see Eq. (8)) a relation which is more general than Heisenberg's uncertainty relation.

This new relation should not reduce to Heisenberg's relation when the particles are correlated.

This generalized uncertainty relation suggests that, in conditional measurements, we can have states where $\Delta Q_1 \Delta P_1 < \frac{\hbar}{2}$. An example is the photon at the virtual slit in the experiment of Kim and Shih. We can even get, at least theoretically, for example, a minimum dispersion for position without a divergence in the momentum dispersion. These possibilities we intend to explore further in the future.

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